Lecture 14. r-th roots & Chiodo's formula

- Twisted curves & r-th root moduli space

- GRR between smooth schemes

- Chiodo's formula

Reference.

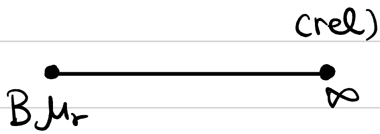
Chiodo
- Towards an enumerative geometry of the moduli space of twisted curves and r th roots
  ↳ GRR calculation
- Stable twisted curves and their r-spin structures
  ↳ Foundation of the moduli construction

Abramovich-Vistoli :
- Compactifying the space of stable maps

§1. Twisted Curves.

<u>Reminder</u> : We used the following geometry to prove Pixton's formula

$\mathbb{P}^1[r]_\infty$:



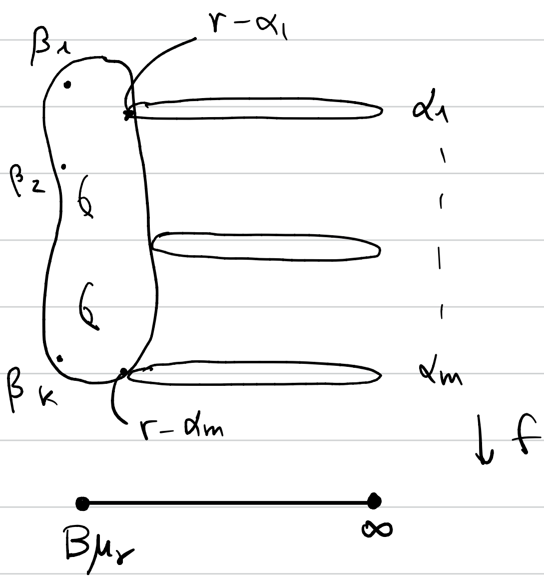
orbifold / relative target.

$\mathbb{C}^* \curvearrowright \mathbb{P}^1[r]_\infty$.

$\boxed{\text{Type } 0}$

$\mu_r = \mathbb{Z}/r\mathbb{Z}$

orbifold GW



. Chiodo's formula deals with a curve mapping to the stacky point $B\mu_r$ of the target.

<u>Q</u> How does this relate to r-th root?

- G-torsors

G = algebraic group (μ_r or $\mathbb{C}^*$).

Def A principal G-bundle (or G-torsor) over X is a space $P \to X$ together with $G \curvearrowright P$ s.t. (étale) locally on X, we have

$$P|_U \cong U \times G .$$

Then there exists a bijection

$$\mathrm{Maps}(X, BG) \longleftrightarrow \{ P \to X : G\text{-torsor} \} .$$

Eg. If $G = \mathbb{C}^*$, $\mathbb{C}^*$-torsor $\Longleftrightarrow$ line bundle

$$P \longmapsto P \times_{\mathbb{C}^*} \mathbb{C}$$

$$L \setminus \{0\} \longleftarrow L$$

μ_r sits inside the following exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \mu_r & \to & \mathbb{C}^* & \longrightarrow & \mathbb{C}^* & \to & 1 \\ & & & & z & \longmapsto & z^r & & \end{array}$$

If P is a μ_r-torsor over a smooth projective curve C, we can take a scalar extension $P \times_{\mu_r} \mathbb{C}^*$ and get a line bundle L_P on C.

$\Rightarrow$ $L_P^{\otimes r} \approx \mathcal{O}_C$. $\qquad\qquad L_P = P \times_{\mu_r} \mathbb{C}$

- What is the issue over $\overline{\mathcal{M}}_g$? $(n=0)$

Now we do this over $\mathcal{M}_g$:

$$\varepsilon : \mathcal{M}_g^{1/r} = \{ (C, L) \mid L^{\otimes r} \cong \mathcal{O}_C \} \longrightarrow \mathcal{M}_g$$

This map is étale, proper & $|\varepsilon^{-1}[C]| = r^{2g}$

Q What happens if we naively extend to $\overline{\mathcal{M}}_g$?

$$\overline{\mathcal{M}}_g^{1/r,\, \text{naive}} = \{ (C, L) \mid L^{\otimes r} \cong \mathcal{O}_C \}.$$

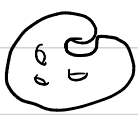
A This is wrong thing to do! (not proper)

(Why?) $\varepsilon : \overline{\mathcal{M}}_g^{1/r,\, \text{naive}} \longrightarrow \overline{\mathcal{M}}_g$ be a morphism forgetting r th root.

We know that ε is étale.

If ε were proper, # of fibers should be constant.

Over $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$, $\# \varepsilon^{-1}(p) = r^{2g}$. Let

$C =$  $\in \partial \overline{\mathcal{M}}_g$.

$$\Rightarrow \# \varepsilon^{-1}[C] = \underbrace{r^{2(g-1)}}_{\substack{r \text{ th roots on} \\ \text{the normalization of } C}} \cdot \underbrace{r}_{\substack{\text{translation function} \\ \text{at the node } \in \mu_r \subset \mathbb{C}^*}} = r^{2g-1} \quad \lightning$$

In general $\varepsilon^{-1}[C] = r^{2g - h^1(C)}$

• Twisted curve & r th root moduli

Solution [Abramovich - Vistoli]

The curve itself should have stacky structure at each node! "twisted curve"

* B : scheme

Def $(\pi : C \longrightarrow B, p_1, \cdots p_n)$ is a family of r-stable curves, or twisted curves, if π is a proper flat morphism (not representable) and

(i) fibers are purely 1-dim'l.

(ii) C^{sm} is an algebraic space

(iii) the coarse moduli $\overline{C} \longrightarrow B$ is a family of stable curves

(iv) (etale) locally at each node:

$$\pi : [U/\mu_r] \longrightarrow T = \operatorname{Spec} A$$

· $U = \operatorname{Spec} A[z,w]/(zw-t)$ $t \in A$

· $\xi . (z,w) = (\xi z, \xi^{-1} w)$, $\xi \in \mu_r$. ← stacky structure at each nodes.

$\rightsquigarrow \overline{\mathcal{M}}_{g,n}(r)$.

Thm [Abramovich–Vistoli, Chiodo]. $\overline{\mathcal{M}}_{g,n}(r)$ is a proper, smooth irreducible DM-stack of $\dim = 3g-3+n$.

$\varepsilon : \overline{\mathcal{M}}_{g,n}(r) \longrightarrow \overline{\mathcal{M}}_{g,n}$ is surjective, finite, flat.

Now we consider r-th roots over twisted curves.

Recall If $X = [M/G]$, a line bundle on $X \Leftrightarrow$ G-equivariant line bundle L on M.

e.g. $X = B\mu_r$, line bundles on $X \Leftrightarrow$ 1-dim μ_r-rep.

$$\xi . z = \xi^i . z, \quad 0 \le i \le r-1.$$

monodromy

Consider the following moduli space

$$\overline{\mathcal{M}}^{1/r}_{g,A} = \{ (C, L) \mid C : r\text{-stable curve}, \ L^{\otimes r} \simeq \mathcal{O}(-\textstyle\sum a_i p_i) \}.$$

Thm. $\overline{\mathcal{M}}^{1/r}_{g,A}$ is a smooth, proper DM-stack of $\dim = 3g-3+n$.

and $\overline{\mathcal{M}}^{1/r}_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}(r)$ is finite & étale.

There is a slightly different compactification of r-th roots:

orbifold stable maps to $B\mu_r$ ($\Leftarrow$ fixed point of Type 0)

$$\overline{\mathcal{M}}_{g,\underset{\sim}{A}}(B\mu_r, 0)$$

$\leftarrow$ orbifold structure at nodes & markings

monodromy at the marking

$\overline{\mathcal{M}}_{g,A}(B\mu_r, 0) = \{ (C, L) \mid$ C : twisted curve. L : line bundle on C.
$L^{\otimes r} \cong \mathcal{O}_C$ & monodromy at $p_i = a_i$ (or $r - a_i$) $\}$.

Q How does $\overline{\mathcal{M}}_{g,A}(B\mu_r, 0)$ recover twisting by a section?

Take $(C, L) \in \overline{\mathcal{M}}_{g,A}(B\mu_r, 0)$ where C is smooth.
Let $\varepsilon : C \longrightarrow \overline{C}$ be the coarse space. Suppose $L^{\otimes r} \cong \mathcal{O}_C$.
Then $\varepsilon_* L$ is a line bundle and

$$(\varepsilon_* L)^{\otimes r} = \mathcal{O}_{\overline{C}}\left(-\sum a_i p_i\right)$$

So we recover the twist on $\overline{C}$. .

Since we have a map $\overline{\mathcal{M}}_{g,A}(B\mu_r, 0) \longrightarrow \overline{\mathcal{M}}_{g,A}^{1/r}$ we can translate Chiodo's result on $\overline{\mathcal{M}}_{g,A}^{1/r}$ to the orbifold stable map space.

• Boundary strata of $\overline{\mathcal{M}}_g^{1/r}$.

Boundary stratum of $\overline{\mathcal{M}}_g^{1/r}$ corresponds to a stable graph Γ and extra data

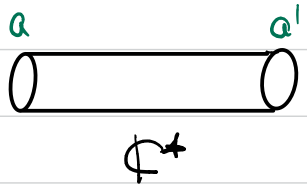
$$w : E(\Gamma) \longrightarrow \{0, 1, \cdots, r-1\}$$

coming from the representation of $\mu_r \circlearrowright L_e$ ← fiber at the node e.

• Balancing condition at $e = (h, h')$

What are the r-th roots of the trivial line bundle over $\mathbb{C}^*$?

$H^1_{et}(\mathbb{C}^*, \mu_r) \approx \mu_r$. (or $\pi_1(B\mu_r) \approx \mu_r$)



$0 \leq a, a' \leq r-1$, $a + a' \equiv 0 \pmod r$

We consider a family of nodal curves $a + a' \equiv 0 \pmod r$

Boundary strata of $\overline{\mathcal{M}}_g^{1/r}$ $\Longleftrightarrow$ stable graph with a weighting mod r.

(more precisely $\overline{\mathcal{M}}_{g,A}(B\mu_r, 0)$)

Let's go back to $\varepsilon : \overline{\mathcal{M}}^{1/r}_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}(r)$:

$|W_{\Gamma,r}| = r^{h^1(\Gamma)} \implies |\varepsilon^{-1}[C]| = r^{2g-h^1(\Gamma)} \cdot r^{h^1(\Gamma)} = r^{2g}$

⊛ The first chern class of the normal bundle at $e=(h,h')$

$= -\frac{\psi_h+\psi_{h'}}{r}$, $\quad \psi_h = \varepsilon^* \psi_h$, $\psi_{h'} = \varepsilon^* \psi_{h'}$,

$$\varepsilon : \overline{\mathcal{M}}^{1/r}_{\Gamma,w} \longrightarrow \overline{\mathcal{M}}_{\Gamma}$$

§2. Grothendieck-Riemann-Roch formula

• Characteristic classes

$V \to X$: rank r vector bundle.

Assume $V \cong L_1 \oplus \cdots \oplus L_r$ splits into line bundles.
(this is not a serious assumption. "Splitting principle")

$\alpha_i = c_1(L_i) \in H^2(X)$ (or $A^1(X)$)

We define several cohomology classes on X in terms of α_i.

(1) Chern classes : $c(V) = \sum_{i=0}^{r} \underbrace{c_i(V)}_{} = \prod_{i=1}^{r}(1+\alpha_i)$

i th elementary symmetric polynomial in α_i

(2) Chern character : $ch(V) = \sum_{i=1}^{r} e^{\alpha_i}$

$$= r + c_1(V) + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3)$$

* If $0 \to E' \to E \to E'' \to 0$ is a short exact seq of vector bundles.

$$ch(E) = ch(E') + ch(E'').$$

* Let X = quasi-projective scheme. $\mathcal{F} \in Coh(X)$. Then

$$0 \to \mathcal{L}_m \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0. \quad \mathcal{L}_i = \text{loc free}$$

$$ch(\mathcal{F}) := \sum (-1)^i ch_i(\mathcal{L}_i)$$ ← independent of the choice of resolution

(3) Todd class : $Td(V) = \prod_{i=1}^{r} Q(\alpha_i)$ where

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_{2k}}{(2k)!} x^{2k}$$

* $Td(E) = Td(E') \, Td(E'')$
* $Td(E)$ is invertible in $H^*(X)$.

• GRR

X, Y : quasi-projective, smooth scheme, $f: X \to Y$: projective
For $\mathcal{F} \in Coh(X)$, $R^i f_* \mathcal{F}$ is a coherent sheaf on Y
associated to a presheaf

$$U \longmapsto H^i(f^{-1}U, \mathcal{F}).$$

$\rightsquigarrow$ $R^\bullet f_* : \underbrace{K_0(X)}_{K_0\text{-group of } X} \longrightarrow K_0(Y)$, $[\mathcal{F}] \longmapsto \sum_i (-1)^i [R^i f_* \mathcal{F}]$

<u>Thm</u> (Grothendieck) If $f: X \to Y$ as above and $\mathcal{F} \in Coh(X)$, then

$$\underbrace{ch(R^\bullet f_* \mathcal{F})}_{\text{derived pushforward}} \cdot Td(T_Y) = \underbrace{f_*}_{\text{pushforward in cohomology (or chow)}} (ch(\mathcal{F}) \cdot Td(T_X))$$

Example $f : X \longrightarrow \mathrm{Spec}(\mathbb{C})$. Then

$$\chi(\mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F}) . \mathrm{Td}(T_X)$$

The GRR naturally extends to a morphism between smooth DM stacks when f is representable (ie f induces $\mathrm{Aut}(x) \hookrightarrow \mathrm{Aut}(f(x))$)

§3. Chiodo's formula

Set up:

universal r th root

$$\begin{tikzcd} & \mathcal{L} \arrow[dl] \\ C^{1/r}_{g,A} \arrow[d, "\pi"'] & \\ \overline{M}^{1/r}_{g,A} & \end{tikzcd}$$

universal curve

Goal : $ch(R\pi_*\mathcal{L})$

We have a technical obstruction to apply GRR to π.

Problem π is not representable. because π is a family of stacky curves. If we apply GRR formula directly, we will get a wrong answer!

($\exists$ GRR for DM-type morphisms, but this is not what we are doing here)

H.H. Tseng

Idea [Chiodo] Let $(C/B, \mathcal{L})$ be a family of r th roots over a smooth base B.

$$\begin{tikzcd} & & \widetilde{\mathcal{L}} \arrow[dl] \\ & \widetilde{C} \arrow[d, "\rho"] \arrow[ddl, bend left=60, "\widetilde{\pi}"] & \\ \mathcal{L} \arrow[d] & & \\ C \arrow[r, "\iota"] \arrow[d, "\pi"'] & \overline{C} \arrow[dl, "\overline{\pi}"] & \\ B & & \end{tikzcd}$$

smooth

$\widetilde{\pi}$ ← family of semistable curves

$\nu : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$: coarse space (basically you kill fiberwise stacky structure)

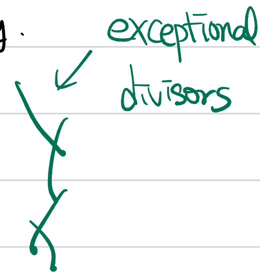
singular space

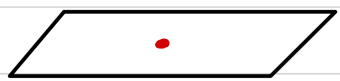
$\rho : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}$: a particular resolution of singularities

⊙ (étale) local picture at nodes : for simplicity, we look at analogous situation for twisted curves

usual family of nodes : $\mathbb{C}^2 \longrightarrow \mathbb{C}$, $(x,y) \longmapsto z = xy$.

$$\begin{tikzcd} & \widetilde{\mathbb{C}^2/\mu_r} \arrow[d, "\rho"] \\ {[\mathbb{C}^2/\mu_r]} \arrow[r, "\nu"] & \mathbb{C}^2/\mu_r \end{tikzcd}$$





· $\mathbb{C}^2/\mu_r$ has a singularity at the origin "A_{r-1}-singularity".
We can blow up the center $\lfloor \frac{r}{2} \rfloor$ times, and get a smooth surface $\widetilde{\mathbb{C}^2/\mu_r}$.

This construction can be done in a family and get $\tilde{\mathcal{C}}$ which is smooth and $\tilde{\pi} : \tilde{\mathcal{C}} \longrightarrow B$ is representable.

(it is no longer stable (only semistable) but it really does not matter)

How does this help us?

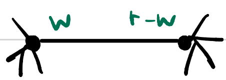
Actually Chiodo proved more. Namely, he constructed a line bundle $\tilde{\mathcal{L}}$ on $\tilde{C}$ s.t

$$ch(R\pi_*\mathcal{L}) = ch(R\tilde{\pi}_*\tilde{\mathcal{L}}) \in H^*(B)$$

We can apply GRR for $\tilde{C}/B$!! It is possible to express $c_1(\tilde{\mathcal{L}})$ in terms of $c_1(\omega_{\tilde{C}/B})$, divisors associated to sections, exceptional divisors.

Theorem (Chiodo)

$$ch_m(R\pi_*\mathcal{L}) = \frac{B_{m+1}}{(m+1)!}\kappa_m - \sum_{i=1}^{n} \frac{B_{m+1}\left(\frac{a_i}{r}\right)}{(m+1)!}\psi_i^m + \frac{r}{2}\sum_{w=0}^{r-1} \frac{B_{m+1}\left(\frac{w}{r}\right)}{(m+1)!}\, \xi_{\Gamma_w *}\left[\frac{\psi_h^m - (-\psi_{h'})^m}{\psi_h + \psi_{h'}}\right]$$

where • $[\Gamma_w] =$ 

• $$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

The proof of Chiodo's formula is rather straight forward. You have to be well-organized to compute Todd classes and multiplication of tautological classes, etc. We will omit the detail. $\boxtimes$

Using the formula

$$c(-E) = \exp\Big(\sum_{m\geq 1} (-1)^m (m-1)!\, ch_m(E)\Big)$$

we get the expression of

$$\varepsilon_* c(-R\pi_* \mathcal{L}) \quad \in H^*(\overline{\mathcal{M}}_{g,n})$$

$\varepsilon : \overline{\mathcal{M}}^{1/r}_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}$.

$\circledast$ We are interested in the leading term in r and most of terms in Chiodo's formula does <u>not</u> contribute.

$\Rightarrow$ Pixton's polynomial $P^{d,r}_{g,A} \in R^d(\overline{\mathcal{M}}_{g,n})$

<u>Exercise</u> We are ready to read the introduction of [JPPZ1]